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ABSTRACT

This publication was prepared to assist school personnel in strengthening their overall mathematics programs. These materials give an overview of a variety of teaching techniques for use in the first eight grades. The illustrative problems are concerned with those topics usually studied at the elementary school level. The problem-solving approaches recommended are designed to be useful in every subject. The materials in this publication may also prove valuable as a basis for in-service teacher education. Chapters include: (1) "An Enlarged View of Problem Solving"; (2) "How to Create Problems"; (3) "Organizing Facts to Perceive Patterns"; (4) "Some More Attempts at Creating Problems"; (5) "Many, Many, Many"; and (6) "Some Concluding Thoughts on Problem Solving." Appended is a section on students with handicapping conditions. (KR)

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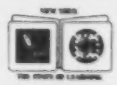

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FOREWORD

This publication has been prepared to assist school personnel in strengthening their overall mathematics programs. These materials do not provide teachers with an in-depth diagnostic instrument. However, they do give an overview of a variety of interesting teaching techniques. In general, the publication is suitable for use in the first eight grades. The illustrative problems are concerned with those topics usually studied at the elementary school level. The problem-solving approaches recommended in this booklet are so designed that they should be useful in every subject. The materials contained in this publication should also prove valuable as a basis for in-service teacher training.

The first publication was developed under the guidance of a committee of mathematics educators that included:

Stephen Brown - State University of New York at Buffalo

Leroy Callahan - State University of New York at Buffalo

Joseph Gehringer - Junior High School 65, Manhattan

Frank Guerrero, Jr. - Nassau County Board of Cooperative Educational Services

Claire Newman - Queens College

Fredric Paul - New York State Education Department

Anne Peskin - City College of New York

George Ropes - Edgemont Public Schools

John Sullivan - New York State Education Department

Drawing upon his deep interest and long experience in teaching problem solving, Stephen Brown organized the contributions of the other committee members and wrote the first draft. The first draft of the booklet was then reviewed by the following mathematics educators: Hamilton Blum, Jacob Cohen, Agnes Higgins, James Jordy, Robert Larsson, Felix Lebaki, Elaine Mintz, and Harry Ruderman. Joseph Gehringer reviewed the several drafts with great thoroughness and made many valuable suggestions. John Sullivan, Associate in the Bureau of Mathematics Education, coordinated the project and prepared that draft. The publication was prepared for production in the Bureau of Curriculum Development in cooperation with the Bureau of Bilingual Education.

This second publication represents a revision, which was completed by Stephen Brown.

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I.

AN ENLARGED VIEW OF PROBLEM SOLVING

Schools exist for many reasons. One of the most important reasons is to teach children how to solve problems. Mathematics has always been supremely useful in solving problems, and the usefulness of the subject is increasing enormously as our societies become more and more complex.

Problem solving can be exciting, engrossing, and valuable beyond measure. We hope to demonstrate this on the following pages.

An Anecdote

Anecdotes have a way of illuminating important issues. We begin with an anecdote and then use it to sketch some major concerns expressed in this booklet.

Put yourself in the place of a 9-year-old who is just learning long division. Suppose someone gave you the following problem: divide 4266 by 17. We assume that your calculation will look something like the following:

```
   250 R 16
17)4266
   34
    86
    85
     16
.............
      0
     16
```

Suppose, however, that you felt a little shaky about long division and that your insecurities began at the dotted line. You felt uncomfortable about dividing 17 into 16, for you are not accustomed to dividing a bigger number into a smaller one. You would have had a quotient of 25 but would not know how to proceed further. What might you do?

The above dilemma is one that Sharon, an intellectually alive but somewhat timid young lady, experienced recently. She came to an older friend for help with the problem.

Sharon: I'm confused. I can't divide 17 into 16.

Friend: Why not?

Sharon: I don't know.

Friend: Try something.

Sharon: What?

Friend: Do you know anything that might help you?

Sharon: No.

Friend: Take your time.

Sharon: Well, 17 doesn't go into 16, but 17 goes into 17, so I'll do this (places a 1 next to the 25):

$$\begin{array}{r} 251 \\ 17\overline{)4266} \\ \underline{34} \\ 86 \\ \underline{85} \\ 16 \end{array}$$

Friend: Then what?

Sharon: Well, I know my answer's too big.

Friend: What would you do?

Sharon: Since I have 16 instead of 17 as the last number, I have to subtract one as a remainder. I could write:

$$\begin{array}{r} 251\ \text{R}\ (-1) \\ 17\overline{)4266} \end{array}$$

Friend: How would you check it?

Sharon: That's easy. Multiply 251 by 17 and subtract 1 from the answer.

Friend: That's beautiful. I love it! You just invented a new way of doing division that maybe no one ever thought about before.

Sharon: Stop! I don't want to do it that way anymore.

Friend: Why not?

Sharon: Show me the right way.

Friend: What do you mean?

Sharon: Show me the way they want me to do it in school.

Friend: Why?

Sharon: If I do my problems like that in school, the teacher will tell me that it's not right for me to do it that way. The teacher will tell me that maybe when I get older and understand a lot more I can do division that way.

This anecdote presents some important questions. Before reading on, discuss this anecdote with a colleague of yours. What do you think of Sharon's reaction? If Sharon came to you or a colleague of yours with her new way of getting the answer, how would you respond? What are some good reasons for her teacher to behave as Sharon anticipates? What are some bad reasons?*

"After Math"

You may be interested in knowing what we see as good or bad reasons. We approach them indirectly by making several observations with regard to the anecdote:

(1) Sharon solved a problem.

(2) Sharon did something that many youngsters would think is impossible.

Let us look at the second assertion. What did she do that others might have thought impossible? Notice she was working on an algorithm — a mechanical procedure for getting answers — and she got stuck. She couldn't go any further. She might have said what you have heard many of your youngsters say: "I can't do that because I haven't been taught how."

Students who make such remarks are *not* necessarily lazy or stupid. They are saying something about what they believe to be the nature of mathematics. They believe that there are experts—like teachers—who have answers. They believe that experts know both the right questions and the right answers. They think of mathematics as being a collection of rules and algorithms. They believe that if they are taught these rules and algorithms correctly, they will be able to answer any question you give them.

What if you did not show them how to answer a certain question in the past? What if you had never shown them explicitly how to divide a larger number into a smaller one (as in the case of 17 into 16)? Well then, they (so they believe) should not be given that question to answer. It's not fair because you didn't show them how to do it. You, the teacher, did *not* do your job if you expect them to answer a question without having provided them with a procedure for getting the answer.

Give your youngsters the following problem: Suppose there is an athletic game that is played in which a team can score either 3 points or 7 points. With such a set-up, there are some scores that cannot appear on the scoreboard. For example, scores of 2 points or 4 points will not appear, but a team can get a score of 6 or 10 points. What are some scores you cannot get? Is there some point beyond which you can get all scores?

* See "Sharon's 'Kye,'" by Stephen I. Brown (in references at end) for further analysis of this anecdote.

Give the students a few days to think about the problem; then:

(1) Have them meet in groups of three or four to discuss what answers they came up with.

(2) Have them also discuss in these groups whether or not it was fair for you to assign this problem and why.

It is important for you as a teacher to find out which of your youngsters think the problem is unfair. It will take time and it will take patience to help them see that this attitude is shortsighted.

Mathematics and Problems

Look back at the first observation of the previous subsection. We wrote: "Sharon solved a problem." We have pointed out that this is more of an achievement than it appears on the surface. The important point is *not* so much that she solved this problem, *but rather that she had a conception of mathematics that led her to believe it was fair to expect her to do so.*

What do students believe who do not think it's fair? If they should *not* be given problems, what should they be given in a mathematics lesson? Though they may not use such language, many youngsters believe that mathematics is about *doing exercises* and not about *solving problems*. Doing exercises is not only what many youngsters think mathematics is about, but it is also what most textbooks do most of the time. What is the difference between exercises and problems?

In doing an exercise, a youngster follows a known algorithm or procedure for getting an answer to a question. There is little hesitation in his or her attempt to get an answer. There is little "thinking through." *At best* what one does when conceiving a task as an exercise is to search through procedures that have been learned to find those that apply. It is not wrong to do exercises. Such activity is essential if one is to acquire skills. But it is vitally important to understand that there are differences between an exercise and a problem.

How (without actually carrying out the work) do you find the following sums?

(a) $1 + 18 + 5 + 23 + 76 + 10$

(b) $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$

(c) the sum of all the numbers from 1 to 100

Ask yourself which of the above three you would approach as an *exercise* and which as a *problem*. It makes sense to approach (a) as an exercise. To do so, we just employ a known procedure for finding the sum of several numbers. Though (b) might also be approached as an exercise, we should not view (c) in that light especially if we expect to get an answer in a relatively short time and without the help of calculators.

There are many ways of approaching (c) as a *problem*. Later on in this booklet we shall indicate some of them, but we can use this problem even before solving it to see

what it means to have a problem. To say that you see (c) as a problem means that

- you understand what the desired goal is,
- you do not have immediately available a way of attaining that goal,
- it will take some thinking to achieve it.

In this case, you understand what it means to find a sum, but if the only procedure you have at your disposal for attaining it is to add the terms as in the addition algorithm, you cannot hope to get the answer in a reasonable period of time.

Activities of this sort (which is what this booklet is all about) can be used not only for the purpose of explicitly teaching problem-solving strategies, but also to enable you to diagnose what *conception of knowledge* your students have. You may have to tread gently as you discover which of your students consider it to be unfair for you to ask them to solve problems you have not explicitly taught them already. There may be times when it will be necessary for you to "back-off" a little, especially in the case of those students who believe that it is impossible for them to "solve a problem" unless you have taught them to do so. (See book by Perry in references.)

The Source of Problems

The most reasonable questions for you as a teacher to ask at this point might be: Where could I find a selection of good problems? What are the answers to them? How can I help my students learn how to solve problems? As with all questions, these have embedded within them a set of assumptions. What are these assumptions?

Sometimes we find that by staring hard at the assumptions of our questions, rather than by attempting to answer them, we begin to acquire real wisdom. A number of people take it for granted that the source for good problems lies somewhere (a library, a friend, a good book, a mathematician) other than within themselves. At first glance, that is a very reasonable assumption, especially if one does not consider himself or herself to be an expert or an authority on problem solving.

We would like to persuade you that the ability to come up with good problems resides within each of us. Other assumptions embedded in these questions will also be exposed as we begin our exploration of teaching problem solving.

II.

HOW TO CREATE PROBLEMS

We have distinguished between exercises and problems. In this section, we hope to show how *you* can become a source of good problems by creating them on your own. Indeed, we hope that *you* will find as much intellectual excitement in creating problems for your students as they will find in solving them. Perhaps you will eventually want to teach your students some of these strategies for creating problems.

We begin by taking exercises at different levels of sophistication in the elementary grades and turning them into problems.

Case 1: "Money Makes the World Go Around"

Exercise: Compute the total value of 3 nickels and 4 dimes.

Solution: (3 x 5) + (4 x 10) = 55 Answer: 55 cents

How might you transform the exercise into a problem? One good way would be to *focus backwards*. For many students, finding the *total* value of a handful of coins is a simple exercise. Though they may make many mistakes in arithmetic, the task is conceptually very simple. Focus *now* on the 55 cents instead of the coins, and ask some questions. Here are a number of questions you might ask:

Question 1: Could 55 cents be made up only of nickels?

Question 2: Could 55 cents be made up of dimes?

So far, so simple, though for *some* youngsters questions 1 and 2 would constitute problems. Can you find ways of creating a more challenging question involving 55 cents? So far we have looked at nickels and dimes separately. How would you ask a question that might combine them?

Question 3: What other combinations of nickels and dimes can you come up with to get 55 cents?

Now that we have combined nickels and dimes, what other questions might you ask that could place demands on their *interrelationship*? Below is one example.

Question 4: What is the largest number of dimes you could have if you had nickels and dimes that totaled 55 cents?

So far, we have used only nickels and dimes to yield 55 cents. What else might we do to make this a more challenging activity?

Question 5: Suppose you have nickels, dimes, and pennies that add up to 55 cents. Could you have all pennies? All nickels? All dimes?

How can we interrelate the three kinds of coins and come up with additional questions? As in the case of nickels and dimes, you realize that only certain combinations work. Realizing this, you might ask these additional questions:

Question 6: What is the greatest *number* of coins that can make 55 cents?

Question 7: What is the least *number* of coins that can make 55 cents?

Question 8: Is there any number of coins between the smallest and largest number that will *not* make 55 cents?

There are, of course, many more challenging questions that you could devise with Case 1 as a starting point. It is possible to begin with something relatively routine and work up to challenging questions. Strategies for solving this problem, and other types of interesting questions that could come out of it, will be considered in section III.

Here is another exercise that we can transform into interesting problems.

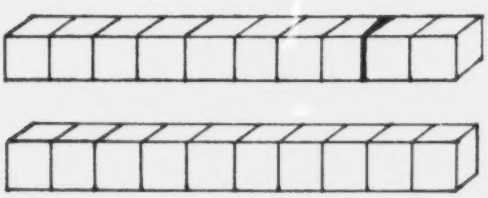
Case 2: "Chickens and Cows"

Exercise: Mr. Brown has three cows and eight chickens. How many legs are there all together?

At this point you should try to compose a few good problems using this basic barnyard setting. Look back at Case 1 for similarities.

Case 3: Some Work with Rods

Our work with algorithms aims to provide us with efficient ways to add, multiply, subtract, and divide numbers. In the lower grades, children frequently make use of concrete material. For example, youngsters persuade themselves that 8 + 2 = 10 by showing that a rod of length 8 joined to one of length 2 has the same length as a rod of length 10.



Suppose you employ the strategy suggested in Case 1 and Case 2, however. *Focus backwards*. Instead of looking first at the *components* of a sum, focus on the sum. Consider the following question: Which pairs will join to match the 10 rod? Clearly, the 2 and 8 rods will work. What else will work? We might be interested not only in those pairs that work but in the *numbers* of pairs. The following question seems reasonable: How many different pairs of rods can we find to match the 10 rod?

Recall that in Case 1 we moved from nickels and dimes to include other denominations. How would that suggest a reasonable question, still maintaining the 10 rod as a focus? There are many possibilities for you to explore.

Case 4: Odd or Even?

We have all memorized multiplication tables as youngsters. The "two times" table for example, begins, "2, 4, 6, 8, 10, 12, 14, 16..."

A *reversal* of the viewpoint enables us to come up with many questions—some of which appear in every text and others of which await *your* insight and creativity to be asked. Let us start with an easy one:

> Given any number, such as 57 832, does it belong to the "two times" table?

You can come up with *many* interesting additional questions.

> Suppose someone has written a five-digit number that allegedly belongs to the "two times" table. Part of it was erased. How much of the number would have to be left intact for me to agree that the number belongs to that table?

We know that *not* all numbers are even (belonging to the "two times" table). That should suggest many questions that might be challenging for some people who are first beginning to think about even numbers. For example:

> If you add two even numbers, what do you get?
>
> If you add two odd numbers, what do you get?

Ask four more questions based upon even and odd numbers in combination with some operations of arithmetic. The best questions are those whose answers you do not know.

Summary

Though we have done a number of things in this section in order to suggest how problems can be *created*, we have for the most part focused on one overall strategy — that of reversing our focus. Instead of taking what was "given" for the purpose of reaching a conclusion, we have taken the *conclusion* and wondered about what might have gotten us there. It was easy to figure out that if we are given a certain number of coins of different denominations, then we would have a certain amount of money. The reverse focus, however, led to a collection of interesting problems. If we are given a certain amount of money, what kinds of coins might we have? Do you see how the concept of reversing what we are given and what we can conclude leads to a whole new set of interesting problems in each of the four cases? We have not yet gone very far in *creating* problems, but we have made a start. In the rest of this booklet, we shall find other strategies for *creating* and for *solving* problems.

III.

ORGANIZING FACTS TO PERCEIVE PATTERNS

Much of problem solving involves efforts to bring order from chaos. Though we are never guaranteed that a particular procedure will unravel the chaos, there are some strategies that may help. Organizing data by use of tables or graphs and searching for patterns in the data are helpful *heuristics* in problem solving. A heuristic is a procedure which, if followed, may be helpful, but which (unlike an algorithm) does *not* assure an answer. Let us return to some of the questions you and we raised in section II in order to indicate the type of organization that is helpful in solving problems.

Using a Table To Organize Data

Let us recall the exercise in Case 2 of Section II. Mr. Brown has three cows and eight chickens. How many legs are there all together? (Answer: 28, not counting Mr. Brown's legs.) Let's make use of reversal as described in Section II.

Question: Suppose there are 28 legs. Does that mean there have to be three cows and eight chickens? How many of each might there be?

How should we approach this new problem? How is it like the first one? (four legs per cow, two per chicken, 28 legs altogether) Could there be one cow? Two cows? More cows? How can we find out?

What is the greatest number of cows possible? The greatest number of chickens? Let's record our information in a *table* so that it's more orderly.

28 legs

number of cows	number of chickens
1	12
2	10
3	8

Now what do you notice? Is there a pattern? Why is it that as the number of cows goes up, the number of chickens goes down? *Use the pattern to predict the number of cows if there are two chickens.*

Children engaged in such problem solving not only attempt to solve this particular problem but also do a tremendous amount of computation along the way. This is an excellent way to practice computation.

Glance once more at Case 1 in section II. By focusing on the *sum* of 55 cents rather than on the specific components given, we come up with a number of interesting problems.

Question: What different combinations of nickels and dimes make 55 cents?

55 cents	
number of nickels	number of dimes
1	5
3	4
5	3

If you ask the students to make predictions about other entries, what might you expect them to say? A few ideas that might emerge (in their own language) are:

- There always seems to be an odd number of nickels.
- If you add the number of nickels and the corresponding number of dimes, you get an interesting pattern.
- This is similar in some ways and different in others from the cows and chickens table.

Notice something interesting here? As we begin to organize data, we frequently begin to make connections and gain insights that are related to other problems that we have seen before.

Seeing Patterns

Let us continue with some of the other questions of Case 1 in Section II. In doing so, we will combine our efforts at being systematic through use of *tables* and other organizational schemes, in an effort to see *patterns* that emerge. Suppose we allow for pennies, nickels, and dimes. Let us now look at questions 6, 7, and 8 of Case 1 in Section II. What is the largest (or smallest) number of coins we can have? Are there numbers of coins in between that we cannot have?

Again, a table in which we attempt to organize the data in a systematic way would help.

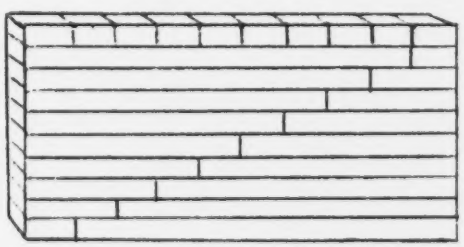
COINS MAKING A TOTAL OF 55 CENTS			
number of pennies	number of nickels	number of dimes	Total number of coins
55	0	0	55
50	1	0	51
45	2	0	47
45	0	1	46
40	3	0	43
(We have not included all possible entries.)			
0	11	0	11
0	9	1	10
0	7	2	9
0	1	5	6

The column on the right reveals the information we are after. It seems clear that the largest number of coins is 55 and the smallest number is six. That part could probably be done by most third graders without a table. What a table helps with, however, is the answer to Question 8 from Case 1 in Section II. What numbers between the largest number of coins (55) and the smallest number of coins (6) cannot be used to make 55 cents?

As in the case of the previous table, having the data organized might well encourage us to see or ask questions we had not seen before. For example, some questions that might emerge (again, in the youngsters' own language) are:

- How many successive whole numbers can you get in a row in the right-hand column of the above table before you have to skip numbers?
- What's the highest number of successions you can get?
- Is there a pattern to the numbers that do not appear in the right-hand column of the table?

In Case 3 in Section II, we started with a very elementary kind of activity that a youngster in the first grade could appreciate. Given a 10-rod, what other pairs of rods have the same length when combined? How many of them are there? Let's *sketch* the results; it usually helps to draw a sketch:



There are nine ways of matching the 10-rod by using others in pairs. Some youngsters may claim that that's too many because the (3,7) pair and the (8,2) pair are used twice. If we wanted to allow the use of a pair only once, then we would have only five pairs for a 10-rod.

Having answered this question, your students might very well ask a question about a modified situation. Suppose we had the same question with an 8- or a 7-rod. What then would be the number of distinct pairs? A table would be useful to record the data we have collected.

Rod Size	Number of Distinct Pairs
10	5
9	4
8	4
7	3
6	3

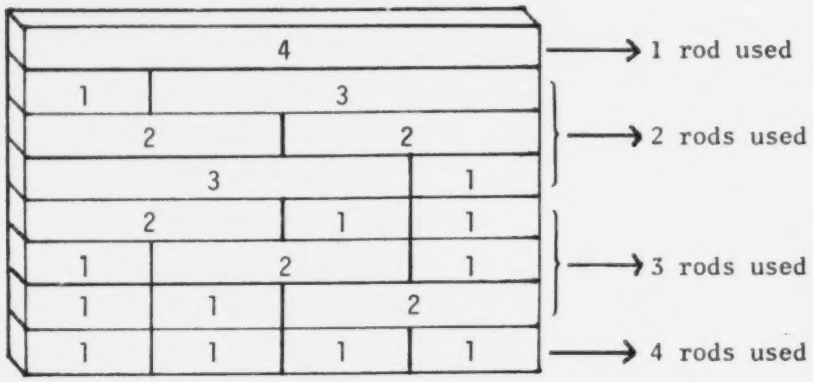
Once more it is profitable to go beyond the original question. The table of rod size and number of distinct pairs may stimulate students to find a general relationship between entries in the two columns. Notice that if the rod is of length 10, there are five distinct pairs; if length 8, there are four distinct pairs; if length 6, there are three distinct pairs. It seems obvious that under *some* circumstances, if the rod is n units long, then there are $1/2 \cdot n$ distinct pairs of rods that might compose it. When does that happen? It seems to work only if n is even (as is the case for 10, 8, 6 as lengths). What if n is odd, however? If we have a rod of length 9, for example, there are four distinct pairs that compose it. How do 4 and 9 relate? Four is *almost* half of 9, but we need to subtract an extra 1/2 to get 4 as an answer. Similarly 3 is almost half of 7, but again, what must be subtracted from half of 7 to get 3 as an answer? There are many ways of seeing the relationship between the length of an odd length and the number of distinct pairs that compose it. One way would be to say that if n is odd, then the number of distinct pairs is 1/2 (n-1). Do you see other ways of expressing the relationship?

It would be interesting and important for you to listen to other ways youngsters have of expressing the relationship. Perhaps some will try to combine both results in one formula; others will have no idea of what you mean by a formula. All of this is fine, but so far not too challenging for youngsters in the upper grades. How might we transform this into an even more challenging question? A glance at what we looked for in the first place might suggest a reasonable expansion of the question. Notice that we allowed only for *pairs* of rods to make a rod of length 10. We know that the rod of length 10, however, could have been made using the following *four* rods: 1, 2, 3, and 4. Of course, once we allow more than two rods, there appears to be a very large number of possibilities. For example, 1 + 1 + 1 + 2 + 1 + 1 + 3 will give us 10, but so will 1 + 2 + 1 + 1 + 3 + 1 + 1.

It looks as if there are so many possibilities that we hardly know how to begin. You might use a *heuristic* that is used frequently in problem solving. Try to make and solve a simpler problem, and then use the knowledge gained to solve the harder one.

Solving a Simpler Problem

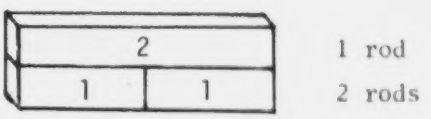
The 10-rod problem (generalized from pairs that compose it, to any number of rods that compose it) is causing a lot of difficulty, because we are awed by the possibilities. It is quite clear here what a simpler case might be. Suppose we had a 4-rod problem instead of a 10-rod problem to examine. The picture below suggests what youngsters might do:



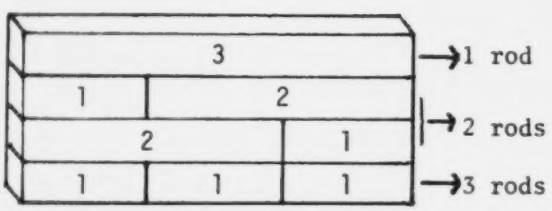
We have drawn a picture so as to convey how many ways there are of arranging one rod, two rods, three rods, and four rods to get a rod of length 4. In this case, *unlike* the previous one, we are calling a combination of rods *different* if their order is different even if their lengths are the same.

What do we have so far in the simpler case of the 4-rod problem (a rod of length 4)? We know it can be represented by one rod in *one* way; two rods in *three* ways; three rods in *three* ways, and four rods in *one* way. We then have *eight* different arrangements for a 4-rod. Does this simpler case shed light on the 10-rod problem — the more complicated one that interested us above? Perhaps you or your students would like to guess at a general relationship of length of rod to number of different arrangements, based upon the case we have just explored.

Appreciating that it is risky business generally to assume a relationship based upon an examination of one case, you might try a few other simple ones. Let us work with the rod of length 2 and that of length 3. Consider the pictures below:



1 rod
2 rods



For a rod of length 2, there is *one* way of using one rod and *one* way of using two rods; therefore, *two* ways altogether.

For a rod of length 3, there is *one* way of using one rod; *two* ways of using two rods; *one* way of using three rods; therefore, *four* ways altogether. Let us now make up a table to summarize results.

Length of Rod	Number of Different Combinations of Rods To Yield It
4	8
3	4
2	2

Do you see a pattern emerging now? Let us choose a special case that might otherwise go unnoticed. If we have a rod of length 1, how many different arrangements will make it up? Obviously, the answer is one. So, our pattern is: 1, 2, 4, 8, for 1, 2, 3, 4 rods respectively. How would you then predict what the case would be for a rod of length 5? Of length 6? Of length 10?

Organizing Things Differently

Based upon the pattern in the last table, you and your students most likely have been able to predict something for the rod of length 10. Though it is reasonable to guess that the table continues in a *doubling* pattern (for length 2, you have two ways; for length 3 you have 2 x 2 ways; for length 4 you have 2 x 2 x 2 ways), it could conceivably break away from that pattern as we progress towards a rod of length 10. One way of gaining confidence in your guess would be to try to look at either the problem or the data a little differently. You could then see if you end with the same result. We will try to look at the data a little differently.

For the data so far, we used at most *four* rods for any row. How many *arrangements* were there that used one rod, two rods, three rods, four rods? That depends upon the size of the rod we began with. Can you or your youngsters suggest methods of recording this data in a table? Below is one attempt to do so.

Size of Rod We	Number of Arrangements Using				
Are Matching	one rod	two rods	three rods	four rods	five rods
1	1				
2	1	1			
3	1	2	1		
4	1	3	3	1	
5	1	?	6	?	?
6	?	?	?	?	?

We have not used the 5-rod. Can you predict how to replace the ?'s with the numbers that belong in those places in the table? Try to match a 5-rod and see if you are correct. If we wanted to know how many arrangements of two rods make up the 5-rod, we are really asking for as many different solutions to the open sentence 5 = ? + ? as we can find.

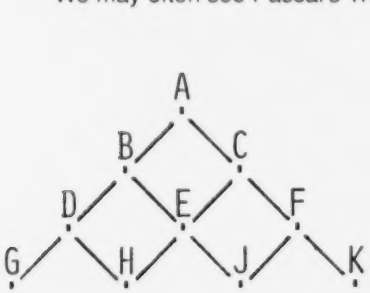
The following is what we obtain:

$5 = 1 + 4$
$5 = 2 + 3$
$5 = 3 + 2$
$5 = 4 + 1$

Therefore, the first "?" in the fifth row of the table above is 4. Does that help you guess the other entries? Can you guess how the row for a 6-rod would be filled in?

The triangular array of numbers that emerges from the table is amazingly useful in mathematics. It is known as Pascal's Triangle, named after a French mathematician who explored it.

We may often see Pascal's Triangle in this form:



We can think of movement along paths joining the lettered points, with the rule that a move can go only downward. There is one way to get from A to B, two ways to get from A to E, and three ways to get from A to J (ABEJ, ACEJ, ACFJ). This is one explanation for the numbers in the triangular array at the right. Replace the question marks in the rod arrangement table above with the numbers that belong in that row. Extend the table until you have 10 rows. Do you see how the rod arrangement table can be transformed into the triangular array? If not, discuss this change with a colleague.

Induction Versus Proof

Let's return to Case 4 from Section II. What happens when we add two even numbers or two odd numbers? There are several ways of approaching the problem. One is by *induction* — examining many special cases searching for patterns. The following examples are instructive:

$2 + 16 = 18$	$17 + 13 = 30$
$14 + 38 = 52$	$21 + 11 = 32$
$16 + 24 = 40$	$43 + 19 = 62$
even + even = ?	odd + odd = ?

We might, on the basis of these examples, believe that it is reasonable to hypothesize:

even + even = even
odd + odd = even

It is possible that we are *wrong* and that we might find two even numbers in the future whose sum is an odd number.

Is there any way we could prove that two evens never have an odd sum? This is a wonderful question to explore with your students.

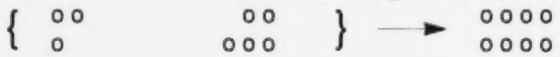
Any even number of dots can be arranged in a rectangular array with two rows:

o o o o
o o o o

Any odd number of dots, if arranged in an array with two rows, will have one dot left over:

o o o
o o

But two odd numbers of dots can be rearranged into a rectangular array:



Now do the same thing for even numbers. Show why two evens must be even by making use of a sketch like that above.

Having examined what happens when you add two odds or two evens, you may suggest other similar questions:

- What happens if you add an even and an odd?
- What happens if you multiply two evens?
- What happens if you multiply an odd and an even?
- What happens if you multiply two odds?

Tail-Biting Behavior : Patterns with a Twist

Let us now return briefly to the questions regarding the ability to predict evenness or oddness for any number no matter how large. List a few of the even numbers in order, and let's see if we can find a pattern: 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28...

You might ask your students what patterns seem to help in recognizing whether or not a number is even.

Some may see that the unit's digit must be 0, 2, 4, 6, or 8. Others may not see that much regularity, but they may observe that the last digit must be divisible by 2. Others may see even *more* regularity than that. They may realize not *only* that the last digits are 0, 2, 4, 6, or 8, but that we progress through the "two times" table, and there is a systematic repetition of the units digit. We can depict the situation as follows:

Revealing this "tail-biting" regularity raises some questions (such as, *Why*?). Instead of trying in a frontal way to see why this happens, we perhaps should examine a more basic issue. Why is it that we only need look at the *last* digit to see if a very large number is divisible by 2? Why is it the case that if 2 divides the last digit, it divides the entire number, and if it does *not* divide the last digit, it does not divide the entire number?

To understand what is happening, we have to understand our notation for numbers in general.

What does 376 mean?

What does 592 mean?

Since we are using a base-10 place value, we realize that the above are short-hand symbols for:

$3 \times 100 + 7 \times 10 + 6$
$5 \times 100 + 9 \times 10 + 2$

Any number of hundreds is an even number; any number of tens is an even number. So we can think of any three-digit number as a number of the form: even + even + units digit. For a four-digit number, we could express it as: even + even + even + units digit. Since any sum of even numbers is always even, any number in base 10 can be thought of as a number of the form:

even + units digit.

If the units digit is also even, then the whole number is even. Why? If the units digit is odd, then the whole number is odd. Why?

Checkers, Anyone?

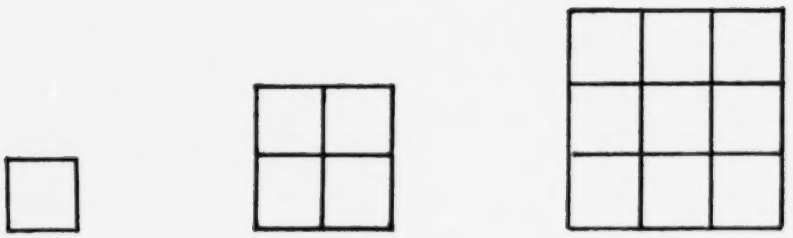
A checkerboard is the setting for many interesting problems. For example: How many squares are there on a checkerboard?

This problem is perfect for practicing several heuristic techniques, particularly:

- Using a simpler problem
- Drawing a sketch
- Making a table
- Perceiving patterns
- Making conjectures
- Checking conjectures

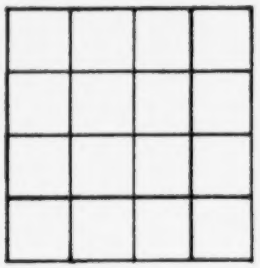
(Try answering the above question before reading on.)

A standard checkerboard has ten squares on each side, which confronts us with a multitude of squares of different sizes. We examine a few simpler variations, such as checkerboards of one square, two squares on a side, three squares on a side.



The checkerboard with two squares on a side has five squares (four little ones and the large one that includes all four). The checkerboard with three squares on a side has 14 squares (four regions with areas of four square units). Do you see a pattern? Any conjectures?

It may be useful to dig deeper and tabulate the numbers of each kind of square. Let's use the "four squares on a side" checkerboard.



There are 16 unit squares, one large square (16 units), four squares of 9 units... The information for a 4 x 4 square is summarized in the fourth row of the table below. We have expanded the table to include similar data for checkerboards of other sizes.

	NUMBER OF SQUARES CONTAINING					
Size of Checkerboard	1 Unit	4 Units	9 Units	16 Units	25 Units	Total No. of Squares
1 x 1						1
2 x 2	4	1				5
3 x 3	9	4	1			14
4 x 4	16	9	4	1		30
5 x 5						

Fill in the boxes for the 5 x 5 checkerboard. Make a conjecture for the 8 x 8 checkerboard. Even if you cannot come up with a precise answer, try to come up with a rough approximation.

Use your experience with this problem to create similar problems.

Summary

In this section, we have described a number of heuristics for solving problems that you have used and will continue to use as you solve problems and as you teach your students to do so. At the beginning, you may wish to have the youngsters engage in the use of these heuristics without making them self-consciously aware that they are doing so. Eventually, however, after you have made significant use of them in your teaching, you will find yourself and your students asking very explicit questions such as:

- Can I illustrate this problem with a sketch?
- Can I think of a simpler example to get me started?
- What is a good way of organizing the data?
- Can I find a pattern in the data?
- Is my answer just a guess or do I know for sure?
- Does this problem have an answer? (Is it possible to do what I am asked to do?)

We have deliberately covered a relatively small number of problems but have done so rather carefully in order to illustrate what we mean by problem solving. In doing so, we have moved *from* a discussion aimed at you as a student *to* one aimed at you as a teacher. We hope you will begin to try out some of these "Cases" with your youngsters even as you find yourself going through the process and even though you may have certain qualms about your ability to control it. In doing so, the youngsters may raise many questions or suggest solutions that puzzle you. We stress that there is no more exciting adventure in teaching problem solving than being puzzled and uncertain yourself. It is important for students to learn that teachers do not know all the answers. In addition to admitting that you may not know some answers, you can reward students for asking good questions. It is worth noting that many of the heuristics for problem solving that we have summarized here are also useful for generating new questions. When it seems appropriate, encourage students to ask new questions using these strategies.

With regard to a search for solutions, however, there are two other heuristics you should make use of: (1) When it is difficult to find a precise answer, search for a good approximation or perhaps upper and lower limits for the answer. (2) Find a colleague in your school or school district who can provide additional insight, if needed.

IV.

SOME MORE ATTEMPTS AT CREATING PROBLEMS

Our focus in section II was explicitly on the creation of problems from exercises. In section III, we developed other problem- posing strategies in the context of focusing on problem solving. In this section, we shall further intertwine problem solving and problem posing as we continue to explore divisibility and introduce the concept of prime number.

On Testing for Divisibility

Perhaps you already knew how to tell quickly whether or not a number is a multiple of 2. For you there was no surprise in the conclusion that the last digit had to be either 0, 2, 4, 6, or 8. One way to infuse such observations with richer meaning would be to try to find a similar result in other cases.

Often a solution of one good problem leads immediately to other interesting problems. Are there numbers other than 2 for which we can test divisibility by looking at the last digit? This is the kind of question that many youngsters will enjoy exploring. There is a simple test for 5 and also for 10. What are these tests? Can you think of any numbers greater than 10 that have such a simple divisibility test? Is there a similar test for 3? Listing multiples of 3 we have:

3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33...

If we look at the units digit here, we discover that every one of the digits, 0 - 9, appears. We would then have no way of telling by looking only at the last digit whether or not a number is a multiple of 3.

The case of divisibility by 3 provides a good opportunity for you to encourage your students to observe, to conjecture, to guess. Since the search for an analogous shortcut to determine divisibility by 3 fails, we could merely give up and pursue other issues. An alternative option is to look for another way of determining by shortcut methods whether or not a number is divisible by 3. With the hope that students might profit as much from being misled as being led, you might suggest that they scrutinize the digits of the numbers. Try multiplying the digits. Examine the differences between digits. Eventually, you will try adding the digits of any number that is a multiple of 3. The table below suggests how your students might make the reasonable conjecture that it is helpful to add the digits of the number and look at the sum. If the sum is divisible by

3, then so is the number. Make sure to explore what happens if the sum is not so divisible.

Multiples of 3	Product of Digits	Sum of Digits
12	2	3
15	5	6
18	8	9
21	2	3
24	8	6
27	14	9
30	0	3

The column at the right shows a pattern that can be used as a test for divisibility by 3. What is the test? What more must you do to insure that the test always works? (Hint: Look at the sum of digits for numbers that are not in the left-hand column).

Now that you have a test for divisibility by 3, you might want to have your students investigate whether or not that kind of test works for any other divisors from 4 to 9.

Bypassing an Impasse: More Cycles

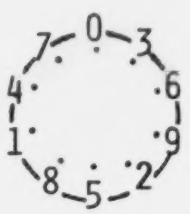
Let us assume for the moment, however, that you and your students have reached an impasse. You are searching for a shortcut to determine if one number is a multiple of another. What might you do? Take the case we just discussed of multiples of 3. Look again at the last digits as we list the "three times" table.

3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39

Though we could *never* use the last digit as a divisibility test for 3, we can see something interesting emerging. Here are the units digits of the multiples of 3.

3, 6, 9, 2, 5, 8, 1, 4, 7, 0, 3, 6, 9, 2, 5...

What do you notice? There is a cycle here. What is it? Just as we had a convenient cycle for the case of last digits of multiples of 2, so we have the following for the units digits of multiples of 3:



- It does not enable us to determine whether or not a number is a multiple of 3, but it does suggest a slew of interesting questions you might now explore. Notice that we were hoping to find a divisibility test for multiples of 3 that is analogous to one for multiples of 2. We failed to find an analogous test, but in deciding to focus upon an important

property that multiples of 2 and of 3 have in common (cycles for units digit), we opened up the potential for new investigation. We have hit upon a regularity that encourages us to ask many other questions. What are some you might ask? Below are two that occur to us:

- What kind of regularity is there for the units digits of numbers in the multiplication table for 4?
- Any number, 0 - 9, may be found as units digits in the multiples of 3, but only 0, 2, 4, 6, 8 may be found as units digits in multiples of 2. What happens in other cases?

Try a few tables of your own, and you will find yourself coming up with many more questions. The "doodling" involved in trying to see what is happening is an activity that will intrigue many of your students. In order to see what is happening, however, it is necessary to organize the information systematically. You might find it worthwhile to explore other numbers. For example, what happens if you look at units digits in squares such as 12, 22, 32, ...? Here are a few:

1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169

Some questions that occur to us and ones that you might want to explore together with your students are:

- Is there a cycle here?
- Is 179 532 a square?
- What is the largest number you can think of, smaller than one million, that you are sure is not a square?
- Try dividing each of the squares by 4. What remainders do you get?
- If you divide by 8, what remainders do you get?
- Choose your own number as a divisor. What remainders do you get?

Some Additional Problems

The purpose of the following problem set is to provide you with an opportunity to test some of the problem-solving and problem-generating ideas presented up to this point.

Problem Set

(1) There is a culture in which all numbers are written backwards. For example, these people write the number 127 958 where we would write it as 859 721. Suppose you are examining a mathematical paper that has no words but only mathematical symbols. You know the paper is trying to say something about divisibility tests however. Which tests of divisibility (for multiples of 2, 3, 5, 10) will hold regardless of the culture the paper comes from? Which will depend upon the culture?

(2) We know divisibility tests for 2, 3, and 5. Below are some questions about how you might create a test for 4. Here are a few multiples of 4: 4, 12, 16, 32, 96

- Does the units digit determine divisibility by 4?
- Does the sum of the digits provide a test for divisibility by 4?
- Which of the following numbers does 4 divide?

32	31
132	131
432	431
1032	1031
57 632	57 631

- Make up a six-digit number ending in 32 that 4 divides.
- Take any two-digit number that 4 divides. Use that in place of "32," and see whether or not 4 divides all the new numbers on the left.
- Take any two-digit number that 4 does not divide, and use that in place of "31." See whether 4 divides all the new numbers on the right.
- Make a conjecture as to what test you would use for divisibility by 4.

(3) The divisibility tests we have suggested assume something that is not at all obvious. They all assume our numbers have been expressed in base 10. You might enjoy exploring what kinds of tests work in other bases. Below are numbers from 1 to 10 represented in base 2, 3, and 5.

Base 10	1	2	3	4	5	6	7	8	9	10
Base 2	1	10	11	100	101	110	111	1000	1001	1010
Base 3	1	2	10	11	12	20	21	22	100	101
Base 5	1	2	3	4	10					

Complete the table for base 5. What are the prime factors of 21 210?

(4) In section III, we made a table to explore the different numbers of coins that could be used to make 55 cents from pennies, nickels, and dimes. It turns out that there are some numbers of coins we could *not* use; for example, 52 through 54 coins could not be used. Suggest a way for creating new coins so that you are able to make 55 cents with any number of coins between the minimum and the maximum number.

(5) Find three consecutive whole numbers such that none of them is a multiple of 3.

(6) Find seven consecutive whole numbers such that none of them is a multiple of 7.

(7) On page 3, we gave you a problem about game scores. We will try to make some sense out of it.

Possible Scores			3			6	7		9	10		12	13	14	15
Not Possible Scores	1	2		4	5			8			11				

It looks as if 11 is the biggest score we cannot get (at least so far). Why is it that once you have *three* successes in a row (12, 13, 14), you will be able to get all higher scores?

(8) Choose two numbers for scores (as in problem 7) such that you will always be able to find a score that *cannot* be gotten.

(9) Look back at problems 5 and 6. In light of the conclusion you probably reached, the problem may seem foolishly worded. Draft an argument in favor of wording the problem as indicated despite its apparent foolishness. Suggest how else you might have worded it.

(10) Problem 3 above was devised by making use of one of the heuristics for creating problems that we discussed in section II. Which one (or ones) might have been used?

(11) Which problems did you find easiest to explore?

(12) How did your enjoyment of the problem relate to the ease of solution?

Questions about Primes

A natural number is prime if it has exactly two different divisors, the number 1 and the number itself. In the case of 11, the divisors are 1 and 11; for 12, however, we have 1, 2, 3, 4, 6, and 12. For convenience we list divisors of the first few numbers:

Number	Divisors of the Number
1	1
2	1,2
3	1,3
4	1,2,4
5	1,5
6	1,2,3,6
7	1,7
8	1,2,4,8
9	1,3,9
10	1,2,5,10
11	1,11
12	1,2,3,4,6,12
13	1,13

Those numbers that have just two divisors are: 2, 3, 5, 7, 11, 13.

The set of prime numbers is a fascinating one. Ideas about primes are interesting and important in the elementary grades. There are many unsolved problems about primes.

We make use of primes when we search for the lowest common multiple or greatest common factor of numbers. An effort to find the same denominator for two fractions for the purpose of adding them involves primes.

Let us indicate briefly some interesting questions about and properties of primes.

- How can I tell if a number is prime?

We have the definition of prime as a natural number that has exactly two divisors. So we do have a procedure for determining if a number is prime. Take the number 91. We try to find divisors other than 1 and 91. Will 2 divide 91? Will 3 divide? If we persist long enough, we find that 7 divides 91 and so does 13. Therefore, 91 is not prime. Since it is a little tedious to carry out such divisions for large numbers, it would be nice if there were a simpler method to identify primes.

A classroom activity that is guaranteed to generate excitement involves a search on the part of youngsters to find a pattern for predicting whether or not a number is prime. If a number has a units digit of 1, 3, or 7, must it be prime?

The numbers 11, 13, 17, 23 fit that description. But 33 and 27 and 21 are not primes!

Student-proposed formulas for prime numbers can be tested with hand-held calculators. A pattern that yields primes at first is 13, 23, 43, 53, 73, 83. When does this pattern break down?

Though there is no readily available formula that will generate prime numbers, there are procedures we can follow that will enable us to identify the primes in a given interval. You may wish to look up the "Sieve of Eratosthenes" as an aid. (See Selected References.*) Regardless of how you do it, you should not have much trouble creating a list of primes between 1 and 300 or so. The list will help you explore some of what follows.

- Unique Factorization

Take the number 42. It is not prime. But you can find primes that multiply together to give 42. Thus $42 = 2 \times 21$ and $2 \times 21 = 2 \times 7 \times 3$.

Any number that is not prime can be written as a product of primes. Thus, $70 = 10 \times 7$ and $10 \times 7 = 5 \times 2 \times 7$. It is *not* surprising that if a number is composite (not prime) it can be written as a product of primes. It is surprising, however, that (except for the order of the factors) there is just one prime factorization of a number. A number may have different factorizations ($12 = 3 \times 4 = 6 \times 2$) but it cannot have different prime factorizations. Problem 10 in the Prime Problem set will persuade you that this is something to be surprised about.

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* Omejc, Eve. "A Different Approach to the Sieve of Eratosthenes," *Arithmetic Teacher*, March 1972, pp. 192-196.

- Goldbach's Conjecture

A man named Goldbach made an interesting observation that most likely came out of "doodling." He knew that if you add any two primes greater than 2 you will always get an even number. Take any two primes (exclude 2) and persuade yourself that their sum would have to be an even number. Progress in the opposite direction is harder. That is, take any even number greater than 2. See if you can find two primes that have that number as a sum. Below are a few examples:

$$4 = 2 + 2$$
$$6 = 3 + 3$$
$$8 = 5 + 3$$
$$10 = 7 + 3$$
$$12 = 7 + 5$$
$$18 = 11 + 7$$

Choose your own even number (less than 100), and see if you can find two primes that add up to it. Goldbach's conjecture is that any even number greater than 2 can be expressed as the sum of two primes. For 200 years no one has disproved this conjecture, nor has it been proved.

Prime Problem Set

1. Goldbach's conjecture has been around for over 250 years and has been neither proven nor disproven in that period of time. There are many problem-generating and problem-solving activities one can create based upon that conjuecture. Since people have tried to prove or disprove the conjecture for such a long time, it most likely involves mathematical ideas that are not obvious ones. Though it might be very difficult to prove (if it is true), it is probably a lot easier to disprove if it is false. What would it take to prove that Goldbach's conjecture is false?

2. How do you think Goldbach came up with his conjecture in the first place? No one really knows the answer to this question, but there is a powerful problem-generation strategy that we discussed in section III that he might have used. Let us look at a reversal of sorts of Goldbach's conjecture

   (a) Look at the set of prime numbers. List the first ten. Which ones are even?

   (b) How many *even* prime numbers are there? (To answer this question, recall the definition of a prime number, and notice what number must divide every even number greater than 2).

   (c) Now, if you add any two prime numbers, what *kind* of number do you get? That question is vague, so let us point to what we are getting at:

   $$2 + 3 = 5$$
   $$7 + 2 = 9$$
   $$5 + 7 = 12$$
   $$13 + 11 = 24$$
   $$19 + 17 = 36$$

   If you include the number 2, then it is possible to add two prime numbers and get an odd number as a sum. If you exclude the number 2, however, then when you add any two prime numbers, what kind of number do you get?

(d) So we now have a *very* simple conjecture, which can be proven by an elementary school child: "If you exclude 2, then the sum of two prime numbers is always even." How would you prove it? Can you prove it without using fancy symbols? Earlier, we spoke about the power of sketching in problem solving. What kind of sketch could enable you to prove this conjecture? (Hint: See the subsection entitled *Induction Versus Proof* in section III).

(e) Now to "psychoanalysis" of Goldbach! How might he have come up with his famous conjecture? We have suggested that by reversing the focus of what is given and what we conclude, we sometimes come up with interesting new problems. Show how Goldbach might have come up with his conjecture by reversing the focus of the conjecture in 2(d).

3. In section III, we discussed a number of heuristics that could be used both in problem solving and in problem generating. One of them was to find a way of organizing data systematically. If you look back at the data used to illustrate Goldbach's conjecture in the text of this section, you will notice that we have left out information between the even numbers 12 and 18. Let us put them in now:

   14 = _____ + _____.
   16 = _____ + _____.

   You are searching for prime numbers that can be inserted in each of the blanks. It is an easy task to find numbers that work, but if you find two colleagues to do this activity with you and if you work at it, the chances are you will be in for a considerable surprise. Notice that there is more than one way of filling in the blanks in each case. In fact, these are the first entries in the search for primes that yield nonunique results. Having discovered that there are *two* ways of representing each of these even numbers as sums of primes, you most likely will be inspired to raise some *new questions*. List two new questions that flow from the discovery that there is more than one answer to the "fill in the blanks" exercise above:

   Question 1: _____

   _____

   Question 2: _____

   _____

4. Try to express *22* as the sum of two primes. As in problem 3 above, find a few colleagues and work in parallel with each other. You should be in for even more of a surprise than in problem 3. What is the surprise? List two new questions that occur to you as a result of the surprise.

   Question 1: _____

   _____

   Question 2: _____

   _____

5. You may at this point find it helpful to summarize a number of the strategies you used in coming up with new questions in problems 1 through 4 above. Notice that one new strategy has evolved in problems 3 and 4 above. That is, you have been inspired to ask questions upon uncovering a surprising circumstance. Another strategy is one that we used as we explored a divisibility test for multiples of 3. We found out that the analogous test for 2 fails, but before moving on to search for a divisibility test for multiples of 3, we noticed that, though it failed, there was an interesting "irrelevant" property: the units digit had an interesting cycle. What was the interesting "irrelevant property" you uncovered in problems 3 and 4 above? Discuss the relationship between problem generation and irrelevant properties.

6. Look at the table of numbers from 1 to 13 in which we searched for divisors of those numbers. We eventually focused upon numbers with exactly *two* different divisors and called those numbers "prime." In the case of the 10-rod problem, we found ourselves generating many new questions once we realized that our focus on *two* rods that could compose 10 units was restricting. If we use that same concept of "challenging the given" in the case of prime numbers, we can come up with many new questions.* That is, the table for divisors of numbers between 1 and 13 indicates that there are *two* numbers with exactly *three* divisors. What numbers are they? Let us call numbers with exactly three different divisors "slime" numbers; it is not hard to make up questions about slimes. Make up two questions about "slime" numbers:

   Question 1: _____

   _____

   Question 2: _____

   _____

7. Having "challenged the given" regarding numbers divisible by exactly two different numbers (in problem 6 above), we are in a position to ask questions not only about "slimes" but about numbers divisible by four or five or six different divisors. Look at the table of divisors of numbers between 1 and 13, and after making a few observations about divisibility by four, five, or six different divisors, come up with three new questions:

   Question 1: _____

   _____

   Question 2: _____

   _____

   Question 3: _____

   _____

* For a development of "challenging the given" as a problem-posing strategy, see Brown, S. I., *Student Generations* and Brown, S. I. and Walter, M. I., *The Art of Problem Posing* (both in bibliography).

8. After doing problem 7 above, find another way of "challenging the given" by asking some questions about prime numbers in the set of fractions. (Assume we are using the same definition of prime as before — that a number is prime if it has exactly two different divisors — but that we are choosing the set of fractions rather than the set of natural numbers as our domain of interest.)

9. Let us challenge the given of domain once more by looking at the set of even numbers: E = 0, 2, 4, 6, 8, 10, 12, 14, 16,... . If a number in *that* set is prime if it has exactly two different divisors, then which numbers are prime? What are the divisors of 2 in E? (Note that 2 does not divide 2 in E, since 1 (the quotient) does not belong to E — in the same way that 2 does not divide 5 in the set of natural numbers since 2 1/2 is not a natural number). What are the divisors of 4 in E? There is only one divisor of 4 in E. What is it? Why? Find the first number in E that has exactly two different divisors. Are there others?

10. After trying problem 9, challenge the given once more by slightly modifying the set E. Let E' = 1, 2, 4, 6, 8, 10, 12, 14, 16,....

    (a) List the primes in E'.

    (b) Can you find a formula for the primes in E'?

    (c) What are the even numbers in E'? (There are a number of different ways of interpreting this question. Suppose we define an even number as one that is divisible by 2. If we choose that definition, show why 6 is not even but 8 is even).

    (d) Factor the following numbers of E' into a product of primes: 8, 12, 18, 30, 36, 72.

    (e) Look back at your answers to the problems in (d) above. In the text of section IV, we mentioned that you should be surprised by the fact that there is unique factorization of primes in the set of natural numbers.** If you have not found the impetus for surprise in (d) above, do the problem (working in parallel) with two or three colleagues and compare your answers. Once you find your analysis in (d) surprising, you ought to be inspired to make up a number of new questions. Do so below:

    Question 1: _____

    _____

    Question 2: _____

    _____

    Question 3: _____

    _____

** For a thorough development of the comparison between conjectures and theorems in the set of natural numbers and in E', see Brown, S.I. *Some "Prime" Comparisons* (in bibliography).

V.
MANY, MANY, MANY

Mathematics is viewed by many students—and by adults as well—as a subject that most people cannot understand; that there is one right answer for every problem; that there is one correct method for arriving at that answer. Good problem-solving training will show that these ideas are myths.

Let's begin by exploring some different ways to solve the same problem.

The Sum from 1 to 100

We come back to a problem that was suggested in section I. What is the sum of the natural numbers from 1 to 100?

With the use of computers or hand-held calculators, this task is not so dreary as it might be if you had to do 99 additions. We are interested primarily in *how* to get an answer, not in the answer itself. Here are some possible approaches:

(a) Break the problem up into smaller components that are more manageable and then look for a pattern. The following is one possible start:

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55$$

Now how about 11 through 20? Here we see that we do *not* need to add up *all* the terms again.

$$11 + 12 + 13 + 14 + 15 + 16 + 17 + 18 + 19 + 20 = \ ?$$

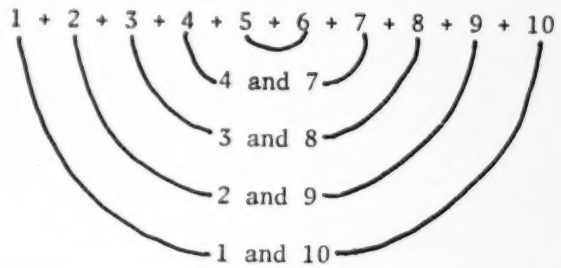
By subtracting 10 from the above calculation, we know that the sum of the *units* digits is 45. Now try 21 through 30. We leave the analysis up to you and your youngsters, but you can see that we have reduced the problem of adding 100 terms to that of adding a much smaller number of terms.

(b) Look at an easier problem. See if there is some pattern in the easier case and then generalize.

Here we might examine what happens if we try to find a simple way to get the sum of numbers from 1 to 10 above.

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$$

These numbers are obviously *not* random. How do they relate? Instead of focusing on every term as if we were reading across the page, consider the whole problem. When adding numbers, we often look for shortcuts. To add 9 + 7 + 1, for example, we might think of (9 + 1) + 7 = 17. This makes us think of grouping the numbers in pairs. The smallest number is 1, and it is counterbalanced by 10, which is the highest. How about 2? What counterbalances it? The following picture comes to mind:



Is there anything constant here? If we add the pairs, we get the same number in each case, 11.

So we have five pairs. What does that come to? Now return to the sum from 1 to 100, and apply the same principle. You will get the answer in a flash.

(c) Try another variation of strategy used in (b).

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$$

One way to achieve balance here is to sense that there is a central "point," an average. What is the central point? It is 5 1/2. As we move one to the right to get 6, we add 1/2 to 5 1/2; as we move one to the left, we subtract 1/2 from 5 1/2 to get 5. What happens as we move two to the right? two to the left? The following diagram is suggested:

Each term can therefore be replaced by 5 1/2 without changing the sum:

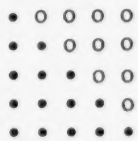
5 1/2 + 5 1/2 + 5 1/2 + 5 1/2 + 5 1/2 + 5 1/2 + 5 1/2 + 5 1/2 + 5 1/2 + 5 1/2

The answer to the above is very easy. Why? Now apply this thinking to get the sum of the numbers from 1 to 100.

(d) Try a geometric approach.

We might think of a triangular array of dots. Instead of the numbers 1 - 10 let us work with 1 - 5:

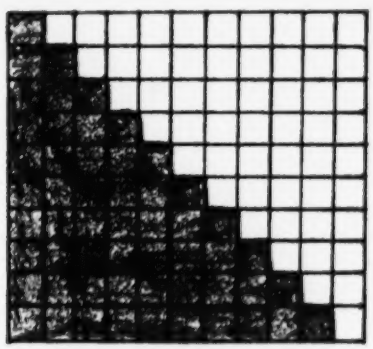
The numbers of dots in the rows correspond to the numbers 1 - 5. After considerable experimentation, we decide to "add" dots to make a rectangular array:



We see that the first four rows form a 5 x 4 rectangular array in which half the dots along with the bottom row constitute the sum we seek. Thus, $\frac{20}{2} + 5 = 15$, which is the sum, $1 + 2 + 3 + 4 + 5$

Have we found a method?

It can be fascinating to analyze the problem with dots, but we leave this for you and instead, use graph paper:



The rows of the shaded portion represent the numbers 1-10.

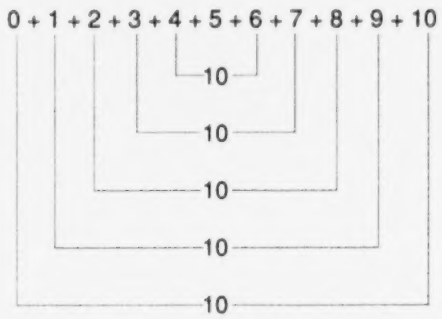
There are many prcductive avenues to follow from here. One good example of geometry supporting problem solving is to "see" the image of the stair-like shaded figure—the unshaded portion to its right. The two figures together form a rectangle with dimensions, 11 x 10 and area 110. This is just twice the number we seek.

Using this knowledge with the larger problem, find the sum of numbers 1 - 100. We visualize a rectangle with dimensions 101 x 100, and its area, 10 100, is exactly twice the sum we are seeking.

(e) One more approach:

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$$

Perhaps you like to count by tens. In the diagram below, there are 5 tens and a 5:



Notice that we add a "silent zero" in order to come up with this scheme.

There are many other ways to find the sum of the numbers from 1 to 100. Let us briefly argue that this problem is not a "fluke." Every problem lends itself to many different solutions. For example, a routine exercise demonstrates this idea, as we see below.

A Subtraction Problem

Look at the following subtraction exercise:

$$\begin{array}{r} 64 \\ -\,28 \end{array}$$

You learned how to "borrow" to get the answer. Are there alternative ways of doing it?

We offer the following two solutions as a way of indicating how even the most routine material can be explored imaginatively.

(a) Try to change it into another problem without the restriction that "borrowing" be used.

What is the problem here? Why do we need to borrow? We need to do so because 8 is greater than 4. Can you think of another problem that is equivalent to this one but in which we do not have the bothersome difference between the two units digits? One way is suggested below:

$$\begin{array}{r} 64 \\ -\,28 \end{array} \qquad\qquad \begin{array}{r} 6? \\ -30 \end{array}$$

Instead of "8" as bottom units digit, a "0" would certainly make life easier. Therefore, change "28" to "30." But if we subtract 30 from a *number* instead of 28, what number must we choose? Can we still use 64? Some very interesting discussion follows this (depending upon the age of your youngsters), but eventually they will agree that:

$$\begin{array}{r} 64 \\ -\,28 \end{array} \quad \text{is equivalent to} \quad \begin{array}{r} 66 \\ -\,30 \end{array}$$

And certainly the one on the right side can be answered without borrowing. This method of subtraction is taught in some places in the world. Let us look at another method of computing 64 - 28.

(b) What is 4 take away 8?

A few years ago, in an elementary school in Weston, Connecticut, a third-grade boy named Kye invented a new algorithm for subtracting. His teacher had been explaining the problem

$$\begin{array}{r} 64 \\ -\ 28 \\ \hline \end{array}$$

and had said: "We can't subtract 8 from 4, so we have to regroup the 60 as..." At this point Kye asked permission to show a different method:

Kye said	Kye wrote
"Oh, yes, you can!	64
Four minus 8 is	- 28
negative 4... and	- 4
60 minus 20 is 40...	40

...and 40 and negative 4 are 36, so the answer is 36."

Kye's teacher listened carefully and worked with Kye and his classmates in trying to decide whether "Kye's method" for subtracting was reliable in general. We need to encourage more children to be like Kye.*

Some Additional Problems

1. Divide your class into halves. For half the youngsters, give them the problem of finding the sum of the numbers from 1 to 30 as an exercise.

   For the other half, tell them the following story: "There was a young boy in grade 4 named Gauss, who eventually became a famous mathematician. His teacher wanted to correct papers at her desk while her students were kept occupied. She gave them the problem of finding the sum of the numbers from 1 to 100. While all of the other students were busily working away, Gauss raised his hand after a few minutes and offered the correct answer." Have students work together and try to figure out how he might have come up with the answer. Compare the results of both groups. Have them discuss with you what accounted for differences in approaches.

2. Find the sum of the following:

   (a) 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1

   (b) Same as above except there are 100 terms to add.

   (c) 2 + 4 + 2 + 4 + 2 + 4 + 2 + 4

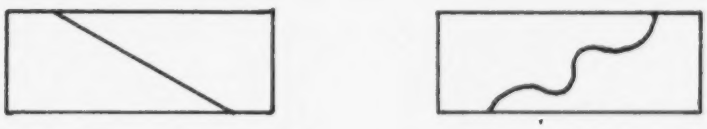
   (d) Same as above except there are 100 terms to add.

* This anecdote is from Robert Davis' article, "The Misuse of Educational Objectives," *Educational Technology*, Nov. '73, pp. 34 - 36.

3. Notice that in the examples to find the sum, we focused on an *even* number of terms. Look back at the solutions to the problems on pages 37-40, a-e, and see how you would analyze the problem if you had an odd number of terms. For example, try finding the sum of the integers from 1 to 75. (If you are stumped here, what heuristics that we have discussed so far might help you get going? Also notice that the analysis using five dots (then forming a rectangle) was different from the one using 10 squares with graph paper. The difference in the analysis depends in part upon evenness and oddness.)

4. We found several solutions to the problem of finding the *sum* of numbers from 1 to 100. Can you find ways of arriving at the following *product*? 1 x 2 x 3 x 4 x 5 x 6 x 7

5. Ask your students to divide a rectangle into two equal parts. Discuss with them why so few did it in an unusual way.



Seeing More Patterns

In section III, we spoke of the role of searching for patterns in both problem solving and problem generating. Here we once again provide some content to encourage both of these activities. In both this subsection and the ones that follow, you will have further opportunities to explore patterns and to debunk the myth that there is only one thing to do or to see in any mathematical activity. Consider the following example. Students were working with magic squares of order 3, using numbers 1 - 9, and had composed these:

(a)

8	1	6
3	5	7
4	9	2

(b)

6	1	8
7	5	3
2	9	4

(c)

2	7	6
9	5	1
4	3	8

(d)

4	3	8
9	5	1
2	7	6

(e)

2	9	4
7	5	3
6	1	8

(f)

4	9	2
3	5	7
8	1	6

(g)

6	7	2
1	5	9
8	3	4

(h)

8	3	4
1	5	9
6	7	2

In a magic square, the sum of the numbers in each column, row, and diagonal is the same. These constitute our data. Students studied the data to detect patterns (regularities). The following questions are connected to some of the patterns they perceived:

- What is the sum of the numbers in each row, column, and diagonal? Why?
- What numbers can occupy the center position? Why?
- What numbers cannot occupy a corner? Why not?

- What numbers cannot be in the same row with 3?
- How are magic squares (a) and (f) alike?
- What are comparable characteristics of magic squares of order 3 that use just even numbers 2 - 18?

Pascal's Triangle Again

```
                                    1
                                  1   1
                                1   2   1
                              1   3   3   1
                            1   4   6   4   1
                          1   5   10  10  5   1
                        1   6   15  20  15  6   1
                      1   7   21  35  35  21  7   1
                    1   8   28  56  70  56  28  8   1
                  1   9   36  84  126 126 84  36  9   1
                1   10  45  120 210 252 210 120 45  10  1
              1   11  45  165 330 462 462 330 165 55  11  1
```

Exploration 1: What kinds of symmetries appear in the table? That is, is there a sense in which the table is repetitious? How so?

Exploration 2: Focus on horizontal rows. What do you observe? For example, what happens if you perform the various operations of arithmetic on the terms of each row?

```
    1   3   3   1

  1   4   6   4   1

1   5   10  10  5   1
```

Exploration 3: Look at the entries along diagonals. For example,

```
      1   1   1   1
    1   2   3   4
  1   3   6   10
1   4   10  20
  1   5   15  35
    1   6   21  56
```

What relationships do you find as you focus on these diagonal columns?

Exploration 4: Look at divisibility properties of numbers along rows, columns, or diagonals. For example, what can you say about even entries? Which rows contain only odd numbers?

Exploration 5: Every counting number must appear somewhere in the table if we extend it far enough. Why? If we eliminate two diagonals, however, this is no longer so. Now look at the triangle with these slanted columns missing and make observations or ask questions about numbers or pairs of numbers that are missing or that are present in the table.

Exploration 6: The following arrays of dots represent a famous sequence of numbers 1, 3, 6, 10, ..., called triangular numbers.

Do you see these triangular numbers in Pascal's Triangle?

Meeting Pascal in Unexpected Places

Just as any given mathematical activity encompasses an unlimited number of avenues to explore, so it is that sometimes activities that appear to be miles apart share unexpected common ground. Let us illustrate this with Pascal's Triangle.

First Meeting

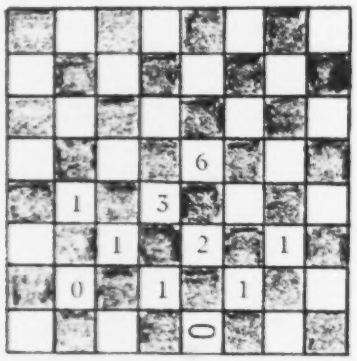
Bob Pascal is a cab driver in Pascalville, a town whose streets all form squares as shown below. When he receives a call at his hack stand (located at AA), he often has a choice of several *direct* routes. From AA to BB, for example, there are two possible routes.

Since a choice of routes helps Bob avoid traffic jams, he decided to make a table showing the number of routes to each point in town from his hack stand at AA. Can you help him complete the table? (We have filled in the "2" in the table below because there are two routes from AA to BB.)

	Second letter is				
	A	B	C	D	E
First letter is A	0	1	1		
First letter is B	1	2			
First letter is C	1				
First letter is D					
First letter is E					

The entries in the table should have a familiar appearance. Where have you seen these before? Can you extend the table? A few days after you have suggested that your students explain this problem, try giving them the following:

Second Meeting



The checker, represented by "○," can move only upward (diagonally), and it must remain on unshaded squares.

- In how many ways can it reach the square containing a 0?
- In how many ways can it reach the square containing a 2?
- In how many ways can it reach the square containing a 3?
- Which are the squares the checker cannot reach? Write a 0 in each.
- Which are the squares the checker can reach in only one way? Write a 1 in each of these squares.
- What is the largest number of ways in which one of the squares can be reached?
- Find those squares that can be reached in 15 ways.
- In each remaining white square, write the number of ways the checker can reach it.

Third Meeting

Suppose you wanted to form a number of different teams from among four students in order to organize a swimming competition.

- How many one-person teams can be formed?
- How many two-person teams can be formed?
- How many three-person teams can be formed?
- How many four-person teams can be formed?

VI.
SOME CONCLUDING THOUGHTS ON PROBLEM SOLVING

Learning to solve problems can easily qualify as one of the most important outcomes of education. Every teacher wants to help students learn to face problems and handle them effectively.

There is much we do not know about teaching problem solving. About some things, though, we feel confident. Students can learn to solve problems not only by solving problems but also by making errors and learning from them, by reflecting on what they have done when they have successfully solved problems, and by learning to make use of helpful heuristics. Help them do it.

We do *not* help students learn problem solving by doing their thinking for them, by showing them exactly what to do, by convincing them that one can do well simply by listening and following directions, or by insisting that they do it our way.

Textbook problems are rarely sufficient to develop problem-solving skills and attitudes. With a single answer sought and an algorithm at their disposal to find it, students are doing drill work. Too much of this may, in fact, inhibit the development of problem-solving ability.

There is no foolproof recipe or method for successful problem solving. The following techniques, though, are helpful, and most of them have been used in this booklet:

- Identify information "wanted," "given," and "needed."
- Use symbols to represent numbers and operations.
- Use open sentences to state conditions of the problem.
- Record data in tables; search for patterns.
- Make guesses; check them.
- Solve part of the problem.
- Solve a similar but simpler problem.
- Come up with an approximation rather than a precise answer.
- Draw diagrams and illustrations of the problem.
- Assume the problem is solved and work backwards.

- Solve the problem in several ways.
- Compose other problems growing out of the solution to a given problem.

The Nature of Questions

The way a teacher asks and answers questions is critically important in teaching problem solving. A good question:

- fosters understanding,
- provides enough direction to get the student moving but not so much as to destroy the problem,
- encourages the student to consider aspects of the problem he or she has not considered.

Questions that ask children to "explore," "investigate," or "look for" are questions that encourage problem solving. Questions that have many answers, or no answer, and questions that lead to other questions all encourage problem solving.

The solutions to many problems have key notions which, when discovered, make the solutions evident. Teachers who are helping students by asking questions and providing hints should, at all costs, avoid giving away the critical idea. Generally, we destroy a problem by giving the student a method of solving it. We should refuse to do students' work for them. This is an essential part of all problem-solving instruction.

An important way to help students become better at problem solving and posing is to provide them with problems or situations that are impossible to do. However, do not tell them the problems are impossible when you make the assignment. We already have come across impossible situations in our discussion of prime numbers. In problem 9 of the section entitled *Prime Problem Set*, for example, one is tempted to ask: List the first few primes in the set of fractions. In fact there are none, but it takes some good thinking to reach that conclusion. In the extension of primes to E': you might ask students to establish that unique factorization holds in that new domain (See Problem 10 in the above-mentioned section).

If you are teaching students geometry, you might ask students to prove that the center of a circle is either inside or outside of a triangle inscribed in the arc (when, in fact, you know that in a right triangle there is a significant surprise here).

The Value of Classic Problems

Some of the best problems are thought up by teachers and students, often by looking at a given situation from several points of view. As we have demonstrated throughout this booklet, the ability to compose good problems is a part of problem solving and needs to be nurtured.

A good problem:

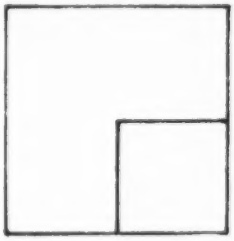
- can be solved in many ways that are within the student's capability.
- intrigues students.

- is readily understandable.
- affords data readily that can be tabulated and analyzed.
- inspires many good similar problems.

Students should realize that problems usually cannot be solved in a few minutes. Often we spend many hours, spaced over days or even weeks, on a good problem, and you should assign some problems that you expect students to work on (reflecting on partial solutions, frustrations, new insights) over a period of a week or two. You can help students by building up a collection of problems that have fascinated each generation of problem solvers. We have presented a few of our favorites on these pages, and some more are included in the next section.

Some of the following problems are not easy. They are offered as grindstones upon which you may sharpen your skills. If too easy, they would not serve that purpose. Some of them can be given to young children without modification, but many will require some fixing up in order to make them understandable to youngsters. At this point, you should have acquired a number of problem-posing skills that will enable you to transform these problems into ones that will entice your students.

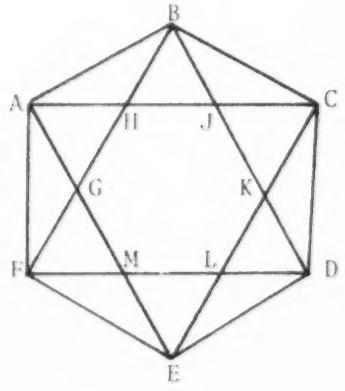
1. A farmer who lives on a square plot of land decides to retire. He wants to keep one-fourth of the land for himself (as shown) and give the rest to his four children. How may the land be divided so that four children may have portions that are the same size and shape?



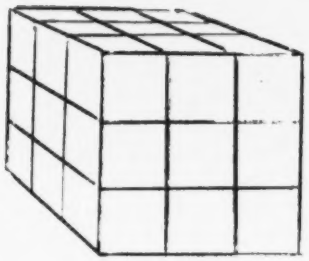
2. Complete the magic square using numbers 1 - 16. Look back at the questions asked on page 28.

	3		13
5		11	
4	15		

3. Express the area of hexagon MLKJHG as a fractional part of the area of regular hexagon ABCDEF.



4. All faces of a cube are painted red, and the cube is then cut into 27 smaller cubes as indicated in the figure at the right.



   (a) How many of the little cubes have:
   - one red face?
   - two red faces?
   - three red faces?
   - four red faces?
   - no red faces?

   (b) If the red cube is cut into 64 little cubes, how many of the little cubes have: (Same questions as in (a) above.)

5. If a line segment is drawn in a rectangle parallel to a side, three rectangles are made: the starting rectangle and two smaller ones. (See Figure 1.) If two such segments are drawn as shown in Figure 2, nine rectangles are formed. Can you see that there are more than four?

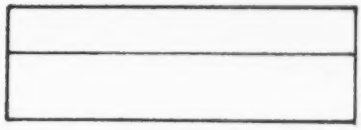


Figure 1

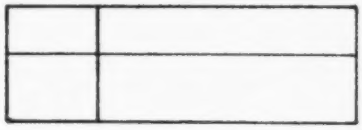


Figure 2

   How many rectangles are formed by three segments? By four? By many?

6. Below is a sentence expressing the sum of two-digit numbers, using each number 0 - 9 as a digit just once:

$$80 + 17 + 36 + 92 + 45 = 270$$

   Compile a set of two-digit numbers whose sum is 72 using each digit 0 - 9 just once. Find a set of two-digit numbers whose sum is 100, using each digit 0 - 9 just once.

7. A fraction may be expressed as the sum of different unit fractions (fractions with 1 as a numerator).

   For example, $\frac{3}{5} = \frac{1}{2} + \frac{1}{10}$

   Express each of the following as the sum of different unit fractions:

$$\frac{2}{5}, \frac{2}{7}, \frac{2}{9}, \frac{5}{6}$$

8. A soda distributor is troubled because his machines often run out of coins to make change; people cannot buy a can of soda if they do not have the right change. The soda costs 35 cents and the machines accept nickels, dimes, and quarters. What recommendations would you make to help the soda distributor?

9. Take any two fractions, say 3/7 and 2/3, and form a third fraction 5/10 by putting the sum of the numerators over the sum of the denominators. Explain why the third

fraction is greater than the smaller of the original two fractions and smaller than the larger.

10. There are five people at a party and each person shakes everyone else's hand. How many handshakes will there be? If there were 50 people, how many handshakes would there be?

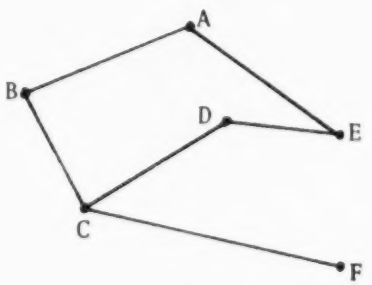
11. A triangle is formed using prime numbers 2 - 23 for the first row. Successive rows are formed using differences of adjacent numbers in the previous row. What patterns do you see?

2 3 5 7 11 13 17 19 23
1 2 2 4 2 4 2 4
1 0 2 2 2 2 2
1 2 0 0 0 0
1 2 0 0 0
1 2 0 0
1 2 0
1 2
1

12. Find a fraction whose decimal representation is 0.111... (the digit 1 repeats endlessly).

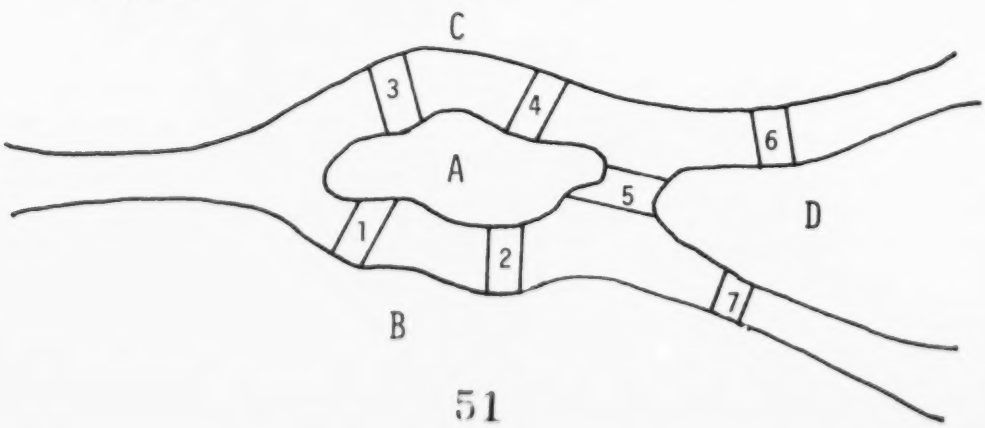
13. Baseball teams in an elimination tournament are eliminated when they lose. If five teams are involved, how many games must be played to determine a winner? Suppose there were 10 teams?

14. The sketch shows the positions of six towns, A, B, C, D, E, F and their connecting road system.



A highway inspector wants to find a way to travel over each section of the road just once. Can you find a way? Could you find a way if there was a road section connecting A and D?

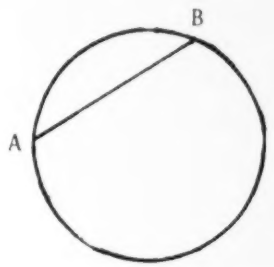
15. This is one of the most famous problems in the history of mathematics. It was solved by the great Swiss mathematician, Leonhard Euler. As the sketch shows, an island, labeled A, is surrounded by branches of a river. There are seven bridges connecting the various territories. Find a route that crosses each bridge once, but only once.



16. The pens in a box are reduced in price from $.50 and are sold for a total of $38.11. What is the sale price and how many pens were in the box?

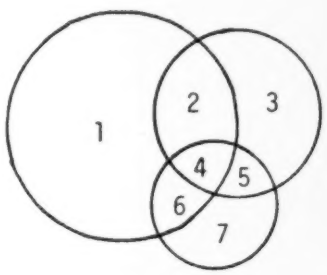
17. An estimate of the product 23 x 38 may be obtained by rounding: 20 x 40 = 800. Make up a rule for estimating the product of two-digit numbers with 5 as units digit, say, 25 x 45. Which gives the best estimate, 20 x 50, 30 x 40, 30 x 50, or some other pair?

18. A line segment joining two points on a circle is a chord as shown. How can we draw the longest possible chord? Draw all possible chords joining three points in a circle. How many chords are there? How many chords are possible joining four points? Five points? Many points?



19. Three intersecting circles form seven nonoverlapping regions.

    What is the largest number of regions formed by seven intersecting circles? By n intersecting circles?



20. What is the sum of the first three odd numbers? The first four? The first 30? The first 100?

21. Find all the whole numbers from 1 to 30 that have factors (ignoring the number itself) whose sum equals the number. (For example, 1 + 2 + 3 = 6.)

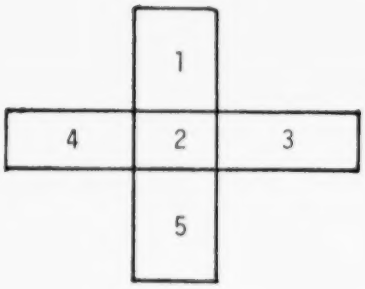
22. How does the sum of the first 100 even numbers compare with the sum of the first 100 odd numbers? How does the sum of the first 100 even numbers compare with the sum of the first 100 natural numbers?

23. Complete the table of square numbers.

Number	1	2	3	4	5	6	7	8	9
Square	1	4	9	16					

Explain why the square of an odd number must be odd and the square of an even number must be even.

24. The maximum number of non-overlapping rectangles that can be formed by two perpendicular intersecting rectangles is five.



Three such intersecting rectangles form 13 nonoverlapping rectangles. How many nonoverlapping rectangles are formed by the intersection of four perpendicular rectangles? Five? Six? Ten? Many?

25. This is a problem that traces back at least to Fibonacci of Italy who wrote of it in 1202. The problem starts with a newly born pair of rabbits. In two months and each month thereafter, this pair begets a new pair, which in turn becomes productive at the end of two months.

A chart shows the increases in the rabbit population:

Month (beginning)	Rabbit Pairs	Number of Pairs
1	P_1	1
2	P_1	1
3	P_1 P_2	2
4	P_1 P_2 P_3	3
5	P_1 P_2 P_3 P_4 P_5	5

At the end of the second month the original pair, P1, produces a new pair, P2.

At the end of the fourth month P1 produces a pair, P4, and P2 produces P5.

What is the size of the rabbit population at the beginning of the sixth month? The seventh month? The twelfth month?

26. These numbers are cubes: 1, 8, 27, 64

They are the cubes of natural numbers, that is:

$1 \times 1 \times 1 = 1$
$2 \times 2 \times 2 = 8$
$3 \times 3 \times 3 = 27$

A mathematics student stated that the difference of any two consecutive cubic numbers is a prime number:

$8 - 1 = 7$
$27 - 8 = 19$
$64 - 27 = 37$

Is the student right?

27. What patterns do you see in this triangle?

$$1$$
$$3 + 5$$
$$7 + 9 + 11$$
$$13 + 15 + 17 + 19$$

28. Explore the patterns in the set of multiples of 12. Invent a procedure for recognizing multiples of 12.

29. A teacher asked Donna to explain how she did a problem such as 91 - 27.

    Donna said: "I can't take 7 from 1 so I'll take 1 from 7 which is 6. Then I take 6 from one of the nine tens, giving me 4. Now I will take two tens from the remaining eight tens to give me six tens. My answer is six tens plus 4 or 64."

    Is this method sound? Why or why not?

30. Devise a method similar to the one explained in the preceding problem to handle problems such as:

$$3\frac{1}{4}$$

$$\begin{array}{r} 3 \\ -\ 1\ 4 \\ \hline \end{array}$$

31. Find three different unit fractions with odd denominators that add up to 2/7.

32. What is the next number in the sequence 2, 3, 5, 8? Justify each of the following as a valid answer: 12, 13, and 2.

33. Multiply a few two-digit numbers by 11. See any patterns? Devise a rule for doing this mentally. Do the same for three-digit numbers multiplied by 11.

34. Here we have an example of addition of two five-digit numbers, but each digit has been replaced by a letter. The problem is to figure out what each letter stands for and recreate the original addition problem.

$$\begin{array}{r} \text{HOCUS} \\ +\ \text{POCUS} \\ \hline \text{PRESTO} \end{array}$$

35. Here is a classic problem of the genre:

$$\begin{array}{r} \text{FORTY} \\ \text{TEN} \\ +\ \text{TEN} \\ \hline \text{SIXTY} \end{array}$$

    Make up a few yourself.

36. Both in problem creating and problem solving, we frequently develop a "psychological set" that limits our options. We assume conditions that are not implied by the problem at all. In problem creating, we frequently don't see the most

fundamental assumptions that are open to modification. Consider the following problem as an example of how a psychological set can limit us:

o o o
o o o
o o o

Draw four straight lines (without lifting the pencil from the paper) that will pass through the nine dots above.

A major difficulty in solving this problem is in the assumption (unwarranted) that there is an imaginary square boundary enclosing the points and that we must remain within the boundary.

- Show how you can solve the problem if you don't make this assumption.
- Give this problem to your students and see if they also make the same assumption.
- Have a discussion with your students on the nature of assumptions they make in solving problems.
- Create one new problem that most people will restrict by making unwarranted assumptions.

APPENDIX
Students with Handicapping Conditions

The Board of Regents, through revising Part 100 Regulations of the Commissioner and the Action Plan, has made a strong commitment to integrating the education of students with handicapping conditions into the total school program. According to Section 100.2(s) "Each student with a handicapping condition, as such term is defined in Section 200.1(ii) of this Chapter, shall have access to the full range of programs and services set forth in this Part to the extent that such programs and services are appropriate to such student's special educational needs." Districts must have policies and procedures in place to make sure that students with handicapping conditions have equal opportunities to access diploma credits, courses, and requirements.

The majority of students with disabilities have the intellectual potential to master the curricula content requirements for a high school diploma. Most students who require special education attend regular education classes in conjunction with specialized instruction and/or related services. These students must attain the same academic standards as their nonhandicapped peers in order to meet these requirements. For this reason, it is very important that at all grade levels students with handicapping conditions receive instruction in the same content areas so as to receive the same informational base that will be required for proficiency on statewide testing programs and diploma requirements.

The teacher providing instruction through this syllabus/ curriculum has the opportunity to provide an educational setting which will enable the students to explore their abilities and interests. Instruction could be provided to students with handicapping conditions either by teachers certified in this subject area or by special education teachers. Teachers certified in this subject area would be providing instruction to students who are recommended by the Committee on Special Education (CSE) as being able to benefit from instruction in a regular educational setting and are appropriately placed in this setting. Special education teachers may also provide this instruction to a class of students with handicapping conditions in a special class setting.

Teachers certified in the subject area should become aware of the needs of those students with handicapping conditions participating in their classes. Instructional techniques and materials must be modified to the extent appropriate to provide students with handicapping conditions the opportunity to meet diploma requirements. Information or assistance is available through special education teachers, administrators, the CSE or a student's Individualized Education Program (IEP).

Additional assistance is available through consultant teacher services. The implementation of this service will allow school districts to provide direct and indirect services to pupils with handicapping conditions who are enrolled full-time in a regular education program. Direct consultant teacher services consist of individualized or group instruction which would provide such students with instructional support in the regular education classroom to help them benefit from their regular education program. Indirect

consultant teacher services will provide support to the regular education teacher in the modification and development of instruction and evaluation that effectively deals with the specialized needs of students with handicapping conditions.

Strategies for Modifying Instructional Techniques and Materials

1. Prior to having a guest speaker or taking field trips, it may be helpful to structure the situation. Use of a checklist or a set of questions generated by the class will help students focus on relevant information. Accessibility for students with handicapping conditions should be considered when field trips are arranged.
2. The use of computer software may be appropriate for activities that require significant amounts of writing by students.
3. Students with handicapping conditions may use alternative testing techniques. The needed testing modifications must be identified in the student's Individualized Education Program. Both special and regular education teachers need to work in close cooperation so that the testing modifications can be used consistently throughout the student's program.
4. Identify, define and preteach key vocabulary. Many terms in this syllabus are specific and may need continuous reinforcement for some students with handicapping conditions. It would also be helpful to provide a list of these key words to the special education teacher in order to provide additional reinforcement in the special educational setting.
5. Check periodically to determine student understanding of lectures, discussions, demonstrations, etc. and how this is related to the overall topic. Encourage students to express their understanding. It may be necessary to have small group discussions or work with partners to determine this.
6. Provide students and special education teachers with a tape of lectures that contain substantial new vocabulary content and of guest speakers for further review within their special education classes.
7. Assign a partner for the duration of a unit to a student as an additional resource to facilitate clarification of daily assignments, timelines for assignments and access to daily class notes.
8. When assigning long-term projects/reports, provide a timeline with benchmarks as indicators for completion of major project/report sections. Students who have difficulty with organizational skills and time sequence may need to see completion of sections to maintain the organization of a lengthy project/report.

Special education teachers providing this instruction must also become familiar with the goals and objectives of the curriculum. It is important that these teachers provide their students with the same or equivalent information contained in the curriculum.

Regardless of who provides the instruction, the cooperation between teachers of regular and of special education programs is essential. It is important for the students as well as the total school environment.

Alternative Testing Techniques

Another consideration in assisting students with handicapping conditions to meet the requirements of regular education is the use of alternative testing techniques. Alternative testing techniques are modifications of testing procedures or formats which provide students with handicapping conditions equal opportunity to participate in testing situations. Such techniques provide the opportunity to demonstrate mastery of skills

and attainment of knowledge without being limited or unfairly restricted by the existence of a handicapping condition.

The Committee on Special Education (CSE) is responsible for identifying and documenting a student's need for alternative testing techniques. This determination is made when a student is initially referred to CSE, is reviewed annually for as long as the student receives special education services, and is reviewed when the student is determined to no longer need special education services. **The modifications are to be used consistently throughout the student's educational program.** Principals ensure that students who have been identified by CSE as educationally handicapped are provided with the alternative testing techniques which have been recommended by CSE and approved by the Board of Education.

Alternative testing techniques which have been specified on student IEPs for use by a student must be used consistently in both special and regular education settings. Regular classroom teachers should be aware of possible alternative testing techniques and should be skilled in their implementation.

The coordination and cooperation of the total school program will assist in providing the opportunity for a greater number of students with a handicapping condition to meet the requirements needed to pursue a high school diploma. The integrated provision of regular education programs, special education programs, remediation, alternative testing techniques, modified teacher techniques and materials, and access to credit through alternatives will assist in enabling such students to pursue high school diplomas to a greater degree. The teacher who provides instruction through this curriculum has a unique opportunity to assist such students in their individual goals.

Additional information on alternative testing modifications is available in the manual entitled *Alternative Techniques for Students with Handicapping Conditions*, which can be obtained from:

New York State Education Department
Office for Education of Children with
Handicapping Conditions
Room 1071 Education Building Annex
Albany, NY 12234

Infusing Awareness of Persons with Disabilities Through Curriculum

In keeping with the concept of integration, the following subgoal of the Action Plan was established:

> *In all subject areas, revisions in the syllabi will include materials and activities related to* generic subgoals such as problem solving, reasoning skills, speaking, capacity to search for information, the use of libraries and increasing student awareness of and *information about the disabled.*

The purpose of this subgoal is to ensure that appropriate activities and materials are available to increase student awareness of disabilities and issues in regard to disabilities.

This curriculum, by design, includes information, activities and materials regarding persons with handicapping conditions. Teachers are encouraged to include other examples as may be appropriate to their classrooms or the situation at hand. Teachers are also encouraged to assess the classroom environment to determine how the environment may contribute to student awareness of persons with disabilities.

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